

Zeta Function Methods and Quantum Fluctuations[‡]

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Abstract. A review of some recent advances in zeta function techniques is given, in problems of pure mathematical nature but also as applied to the computation of quantum vacuum fluctuations in different field theories, and specially with a view to cosmological applications.

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1. Introduction

Zeta function regularization methods are optimally suited for the calculation of the contribution of fluctuations of the vacuum energy, of the quantum fields pervading the universe, to the cosmological constant. Order of magnitude calculations of the absolute contributions of all fields are known to lead to a value which is off by over hundred and twenty orders, as compared with the results obtained from observational fits, what is known as the *new cc problem*. This is difficult to solve and many authors still stick to the old problem to try to prove that basically its value is zero with some perturbations thereof leading to the (small) observed result (Burgess et al., Padmanabhan, etc.) We have also addressed this issue recently in a somewhat similar way, by considering the *additional* contributions to the cosmological constant that may come from the possibly non-trivial topology of space and from specific boundary conditions imposed on braneworld and other seemingly reasonable models that are being considered in the literature (mainly with other purposes too). This kind of Casimir effect would play at a cosmological scale. If the ground value of the cc would be indeed zero (and there are different hints pointing out towards this), we could then be left with this perturbative quantity coming from the topology or boundary conditions and, in particular it could be the fact that the computed number is of the right order of magnitude (and has the right sign, what is also non-trivial) when compared with the observational value. This is proven to be true in some of the aforementioned examples. A further step in this approach would be to consider the so-called dynamical Casimir effect or Davies-Fulling theory. Although there is no clear understanding of how it should be applied in cosmology, some considerations regarding its correct renormalization at laboratory scales have been made recently and we will refer to them later.

The ones above are the physical issues we would like to address ultimately. This needs first the heavy mathematics of zeta functions. They will be presented in the first part of this work in fair detail. The paper is organized as follows, in correspondence with the material presented at the Conference. As a tribute to the actual discoverer of the zeta function, namely Leonhard Euler, in this Celebration Year, Sect. 2 recalls some essential points that lead him to introduce this function—widely considered to be the most important function in Mathematics—with a quick view over the many extensions of that concept in the following centuries. In Sect. 3 we describe how the concept of zeta function of a pseudodifferential operator has become a decisive tool for the regularization of quantum field theories, in special in curved space-time, as clearly realized by S. Hawking. This is exemplified in Sect. 4 through the regularization of the vacuum fluctuations of a quantum system, under some boundary conditions, with a reference to the case of the dynamical Casimir effect (moving boundaries), where regularization issues are particularly involved. Finally, Sect. 5 is devoted to the possible applications of these results in cosmology, concerning the dark energy issue.

2. Euler and the Zeta Function

There are beautiful accounts on how Euler discovered the zeta function (see, e.g. [1, 2]). The harmonic series

$$H = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots \quad (1)$$

was well known to have an infinite sum. Euler asked himself about the ‘prime harmonic series’

$$PH = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \cdots, \quad (2)$$

is it finite or infinite? It is a fact that one cannot split the first series into two, one of them being the second, as

$$\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots\right) + \left(\frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \cdots\right) \quad (3)$$

and try to show that the second is finite (what would mean the first part is infinite). So Euler considered the function

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \cdots \quad (4)$$

Provided s is bigger than 1, one can certainly split it up as

$$\left(1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \cdots\right) + \left(\frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \frac{1}{9^s} + \frac{1}{10^s} + \cdots\right). \quad (5)$$

Now the idea is to prove that when s approaches 1 the first sum becomes divergent. Thus this power s was very useful.

Making things short, a key step in the whole argument is the celebrated factorization of the whole zeta function in terms of prime numbers, namely

$$\zeta(s) = \frac{1}{1 - 1/2^s} \times \frac{1}{1 - 1/3^s} \times \frac{1}{1 - 1/5^s} \times \frac{1}{1 - 1/7^s} \times \frac{1}{1 - 1/11^s} \times \cdots \quad (6)$$

This comes from the fact that for any prime p and any power $s > 1$, setting $x = 1/p^s$ one has the geometric series

$$\frac{1}{1 - 1/p^s} = 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \cdots \quad (7)$$

Euler multiplied together these infinite sums to express his infinite product as a single infinite sum as

$$\frac{1}{p_1^{k_1 s} \cdots p_n^{k_n s}}, \quad (8)$$

with p_1, \dots, p_n primes, k_1, \dots, k_n positive integers, each such combination occurs exactly once and the rhs is just a rearrangement of $\zeta(s)$. It is widely recognized nowadays that Euler’s *infinite product formula* for $\zeta(s)$ marked the beginning of *analytic number theory*.

Dirichlet modified the *zeta function* introduced by Euler. Primes were separated into categories, depending on the *remainder* when divided by k :

$$L(s, \chi) = \frac{\chi(1)}{1^s} + \frac{\chi(2)}{2^s} + \frac{\chi(3)}{3^s} + \frac{\chi(4)}{4^s} + \cdots, \quad (9)$$

where $\chi(n)$ is a special function now known as a Dirichlet ‘*character*’, that splits the primes in the required way. It satisfies the conditions:

- (i) $\chi(mn) = \chi(m)\chi(n)$, for any m, n ;
- (ii) $\chi(n) = \chi(n+k)$, $\forall n$;
- (iii) $\chi(n) = 0$, if n, k have a common factor;
- (iv) $\chi(1) = 1$.

Any function $L(s, \chi)$, where s is a real number bigger than 1 and χ a character, is known as a Dirichlet L -series. The Euler zeta function is the special case with $\chi(n) = 1$ for all n , another example being $\chi(n) = \mu(n)$ (the Möbius function).

A very crucial generalization, introduced by Bernhard Riemann, was to allow s and $\chi(n)$ to be *complex*. The celebrated *Riemann zeta function*, subsequently extended by Hurwitz, Lerch, Epstein, Barnes, etc. increased the number and importance of the zeta function concept decisively. Many results about prime numbers were proven and L -series provide still now a powerful tool for the study of the primes. We should mention for completeness that the concept of zeta function has been yet much more extended, first to the concept of zeta function of a pseudodifferential operator (as we are going to see next), but also to the orbits and trajectories in dynamical systems, under the form of the Selberg zeta function, the Ruelle, the Lefschetz zeta function, and many others that lie outside the scope of this brief summary (Arakelov geometry is one of the most active developments right now). In Ref. [2] a directory of all known zeta functions can be found (there is even one named after the author of the present article, see also Keith Devlin's account there).

3. The Zeta Function of a Pseudodifferential Operator

A *pseudodifferential operator* A of order m on a manifold M_n is defined through its symbol $a(x, \xi)$, which is a function belonging to the space $S^m(\mathbb{R}^n \times \mathbb{R}^n)$ of \mathbb{C}^∞ functions such that for any pair of multi-indexes α, β there exists a constant $C_{\alpha, \beta}$ so that $|\partial_\xi^\alpha \partial_x^\beta a(x, \xi)| \leq C_{\alpha, \beta}(1 + |\xi|)^{m-|\alpha|}$. The definition of A is given, in the distribution sense, by

$$Af(x) = (2\pi)^{-n} \int e^{i\langle x, \xi \rangle} a(x, \xi) \hat{f}(\xi) d\xi, \quad (10)$$

f a smooth function, $f \in \mathcal{S}$, recall $\mathcal{S} = \{f \in C^\infty(\mathbb{R}^n); \sup_x |x^\beta \partial^\alpha f(x)| < \infty, \forall \alpha, \beta \in \mathbb{R}^n\}$, \mathcal{S}' being the space of tempered distributions and \hat{f} the Fourier transform of f . When $a(x, \xi)$ is a polynomial in ξ one gets a differential operator. In general, the order m can be complex. The *symbol* of a Ψ DO has the form $a(x, \xi) = a_m(x, \xi) + a_{m-1}(x, \xi) + \dots + a_{m-j}(x, \xi) + \dots$, being $a_k(x, \xi) = b_k(x) \xi^k$. The symbol $a(x, \xi)$ is said to be *elliptic* if it is invertible for large $|\xi|$ and if there exists a constant C such that $|a(x, \xi)^{-1}| \leq C(1 + |\xi|)^{-m}$, for $|\xi| \geq C$. An elliptic Ψ DO is one with an elliptic symbol.

Pseudodifferential operators [Ψ DO] are useful tools, both in mathematics and in physics. They were crucial for the proof of the uniqueness of the Cauchy problem [3] and also for the proof of the Atiyah-Singer index formula [4]. In quantum field theory they appear in any analytical continuation process (as complex powers of differential

operators, like the Laplacian) [5]. And they constitute nowadays the basic starting point of any rigorous formulation of quantum field theory [6] through microlocalization, a concept that is considered to be the most important step towards the understanding of linear partial differential equations since the invention of distributions [7].

3.1. Definition of the Zeta Function

Let A a positive-definite elliptic Ψ DO of positive order $m \in \mathbb{R}$, acting on the space of smooth sections of E , an n -dimensional vector bundle over M , a closed n -dimensional manifold. The *zeta function* ζ_A is defined as

$$\zeta_A(s) = \text{tr } A^{-s} = \sum_j \lambda_j^{-s}, \quad \text{Re } s > \frac{n}{m} \equiv s_0, \quad (11)$$

where $s_0 = \dim M / \text{ord } A$ is called the *abscissa of convergence* of $\zeta_A(s)$. Under these conditions, it can be proven that $\zeta_A(s)$ has a meromorphic continuation to the whole complex plane \mathbb{C} (regular at $s = 0$), provided that the principal symbol of A (that is $a_m(x, \xi)$) admits a *spectral cut*: $L_\theta = \{\lambda \in \mathbb{C}; \text{Arg } \lambda = \theta, \theta_1 < \theta < \theta_2\}$, $\text{Spec } A \cap L_\theta = \emptyset$ (Agmon-Nirenberg condition). The definition of $\zeta_A(s)$ depends on the position of the cut L_θ . The only possible singularities of $\zeta_A(s)$ are *poles* at $s_k = (n - k)/m$, $k = 0, 1, 2, \dots, n - 1, n + 1, \dots$. M. Kontsevich and S. Vishik have managed to extend this definition to the case when $m \in \mathbb{C}$ (no spectral cut exists) [8].

3.2. Ψ DOs on Boundaryless Manifolds

Let M be a compact n -dim C^∞ manifold without a boundary, E a smooth Hermitian vector bundle over M , A a positive Ψ DO of positive order m in E , elliptic and selfadjoint (admissible). The operator e^{-tA} , namely $e^{-tA} : f \mapsto u$, is the solution operator for the heat equation: $\partial_t u + Au = 0$, with initial value $u|_{t=0} = f$.

This operator is traceclass $\forall t > 0$, and as $t \downarrow 0$ it satisfies

$$\text{tr } e^{-tA} \sim \sum_{j=0}^{\infty} \alpha_j(A) t^{(j-n)/m} + \sum_{k=1}^{\infty} \beta_k(A) t^k \log t. \quad (12)$$

By Mellin transform:

$$\zeta_A(s) = \frac{1}{\Gamma(s)} \int_0^\infty e^{-tA} t^{s-1} dt, \quad (13)$$

$\zeta_A(s)$ has a meromorphic extension with only possible poles at $s_j = (n - j)/m$, $j \in \mathbb{N}$, at most *simple* at $s_j \notin -\mathbb{N}$, and at most *double* at $s_j \in -\mathbb{N}$. Moreover,

$$\alpha_j(A) = \text{Res}_{s=s_j} \Gamma(s) \zeta_A(s), \quad \beta_k(A) = \text{Res}_{s=-k} (s + k) \Gamma(s) \zeta_A(s) \quad (14)$$

The *asymptotic expansion* of the heat kernel determines the *pole structure* of $\zeta_A(s)$, and vice versa. (i) If A is a differential operator, then: $\alpha_j(A) = 0$, j odd, $\beta_k(A) = 0$, $\forall k$. (ii) If $A \geq 0$ one still has the same results, but now for $A - \text{Ker } A$ (subtract Dim Ker to the residue at 0). (iii) If $s_j \in \mathbb{N}$, then $\alpha_j(A)$ is not locally computable [9, 10].

3.3. The Zeta Determinant

Let A a Ψ DO operator with a spectral decomposition: $\{\varphi_i, \lambda_i\}_{i \in I}$, where I is some set of indices. The definition of determinant [12] starts by trying to make sense of the product $\prod_{i \in I} \lambda_i$, which can easily be transformed into a ‘sum’: $\ln \prod_{i \in I} \lambda_i = \sum_{i \in I} \ln \lambda_i$. From the definition of the zeta function of A : $\zeta_A(s) = \sum_{i \in I} \lambda_i^{-s}$, by taking the derivative at $s = 0$: $\zeta'_A(0) = -\sum_{i \in I} \ln \lambda_i$, we arrive at the following definition of determinant of A [11]:

$$\det_\zeta A = \exp[-\zeta'_A(0)]. \quad (15)$$

An older definition (due to Weierstrass) is obtained by subtracting in the series above (when it is such) the leading behavior of λ_i as a function of i , as $i \rightarrow \infty$, until the series $\sum_{i \in I} \ln \lambda_i$ is made to converge. The shortcoming is here—for physical applications—that these additional terms turn out to be *non-local* in general and, thus, they are non-admissible in a renormalization procedure [13].

In algebraic QFT, in order to write down an action in operator language one needs a functional that replaces integration. For the Yang-Mills theory this is the Dixmier trace, which is the *unique* extension of the usual trace to the ideal $\mathcal{L}^{(1,\infty)}$ of the compact operators T such that the partial sums of its spectrum diverge logarithmically as the number of terms in the sum: $\sigma_N(T) \equiv \sum_{j=0}^{N-1} \mu_j = \mathcal{O}(\log N)$, $\mu_0 \geq \mu_1 \geq \dots$. The definition of the Dixmier trace of T is: $\text{Dtr } T = \lim_{N \rightarrow \infty} \frac{1}{\log N} \sigma_N(T)$, provided that the Cesaro means $M(\sigma)(N)$ of the sequence in N are convergent as $N \rightarrow \infty$ [remember that: $M(f)(\lambda) = \frac{1}{\ln \lambda} \int_1^\lambda f(u) \frac{du}{u}$]. Then, the Hardy-Littlewood theorem can be stated in a way that connects the Dixmier trace with the residue of the zeta function of the operator T^{-1} at $s = 1$ (see Connes [14]): $\text{Dtr } T = \lim_{s \rightarrow 1+} (s-1) \zeta_{T^{-1}}(s)$.

3.4. The Wodzicki Residue

The Wodzicki (or noncommutative) residue [15] is the *only* extension of the Dixmier trace to the Ψ DOs which are not in $\mathcal{L}^{(1,\infty)}$. It is the *only* trace one can define in the algebra of Ψ DOs (up to a multiplicative constant), its definition being: $\text{res } A = 2 \text{Res}_{s=0} \text{tr}(A \Delta^{-s})$, with Δ the Laplacian. It satisfies the trace condition: $\text{res}(AB) = \text{res}(BA)$. A very important property is that it can be expressed as an integral (local form) $\text{res } A = \int_{S^*M} \text{tr } a_{-n}(x, \xi) d\xi$ with $S^*M \subset T^*M$ the co-sphere bundle on M (some authors put a coefficient in front of the integral: Adler-Manin residue).

If $\dim M = n = -\text{ord } A$ (M compact Riemann, A elliptic, $n \in \mathbb{N}$) it coincides with the Dixmier trace, and one has $\text{Res}_{s=1} \zeta_A(s) = \frac{1}{n} \text{res } A^{-1}$. The Wodzicki residue continues to make sense for Ψ DOs of arbitrary order and, even if the symbols $a_j(x, \xi)$, $j < m$, are not invariant under coordinate choice, their integral is, and defines a trace. All residua at poles of the zeta function of a Ψ DO can be easily obtained from the Wodzicki residue [16].

3.5. Singularities of ζ_A

A complete determination of the meromorphic structure of some zeta functions in the complex plane can be also obtained by means of the Dixmier trace and the Wodzicki residue. Missing for the full description of the singularities in the above are just the *residua* of all the poles. As for the regular part of the analytic continuation, specific methods have to be used (see later). It can be proven that, under the conditions of existence of the zeta function of A , given above, and being the symbol $a(x, \xi)$ of the operator A analytic in ξ^{-1} at $\xi^{-1} = 0$, then it follows that

$$\text{Res}_{s=s_k} \zeta_A(s) = \frac{1}{m} \text{res } A^{-s_k} = \frac{1}{m} \int_{S^*M} \text{tr } a_{-n}^{-s_k}(x, \xi) d^{n-1}\xi. \quad (16)$$

The proof is rather simple and it can be obtained by invoking the homogeneous component of degree $-n$ of the corresponding power of the principal symbol of A , obtained by the appropriate derivative of a power of the symbol with respect to ξ^{-1} at $\xi^{-1} = 0$, namely

$$a_{-n}^{-s_k}(x, \xi) = \left(\frac{\partial}{\partial \xi^{-1}} \right)^k [\xi^{n-k} a^{(k-n)/m}(x, \xi)] \Big|_{\xi^{-1}=0} \xi^{-n}. \quad (17)$$

Then the proof follows constructively, by easy algebraic manipulation.

3.6. The Multiplicative Anomaly and its Implications

Given A , B and AB Ψ DOs, even if ζ_A , ζ_B and ζ_{AB} exist, it turns out that, in general, $\det_\zeta(AB) \neq \det_\zeta A \det_\zeta B$. The multiplicative (or noncommutative) anomaly (or defect) is defined as:

$$\delta(A, B) = \ln \left[\frac{\det_\zeta(AB)}{\det_\zeta A \det_\zeta B} \right] = -\zeta'_{AB}(0) + \zeta'_A(0) + \zeta'_B(0). \quad (18)$$

Wodzicki's formula for the multiplicative anomaly [15, 17, 18]:

$$\delta(A, B) = \frac{\text{res } \{[\ln \sigma(A, B)]^2\}}{2 \text{ord } A \text{ord } B (\text{ord } A + \text{ord } B)}, \quad \sigma(A, B) := A^{\text{ord } B} B^{-\text{ord } A}. \quad (19)$$

At the level of Quantum Mechanics (QM), where it was originally introduced by Feynman, the path-integral approach is just an alternative formulation of the theory. In QFT it is much more than this, being in many occasions *the* actual formulation of QFT [19]. In short, consider the Gaussian functional integration

$$\int [d\Phi] \exp \left\{ - \int d^D x [\Phi^\dagger(x) (\quad) \Phi(x) + \dots] \right\} \longrightarrow \det (\quad)^\pm, \quad (20)$$

and assume that the operator matrix has the following structure (being each A_i an operator):

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \longrightarrow \begin{pmatrix} A & \\ & B \end{pmatrix}, \quad (21)$$

where the last expression is the result of diagonalizing the operator matrix. A question now arises. What is the determinant of the operator matrix: $\det(AB)$ or $\det A \cdot \det B$? This issue has been very much on discussion [20, 21].

It is difficult to give a general answer to this question, that is, if it is possible to give a universal rule on how to choose the right prescription, and if one can do so on mathematical grounds only, without invoking any physical arguments. To start, we should not forget that the issue at hand at this level is *regularization*. This means, for one, that there may well be different regularized answers that lead, after the corresponding renormalization prescription in each case, to the same renormalized, physically meaningful result. But the renormalization process will generically mean entering into the physics of the problem in order to choose the right criterion. Thus, the answer can in general only be given for the particular example considered. There is no space here in order to enter into a more detailed discussion [20, 21].

Let us just summarize by pointing out the following. First, that a number of serious mistakes and wrong results have appeared in the literature because of forgetting about the multiplicative anomaly. Second, that the Wodzicki formula provides a very convenient and precise way to calculate the anomaly. Third, that this anomaly turns often to be physically meaningful, since it usually (but of course not always) happens that the two different regularized results obtained do indeed lead to two different results after renormalization [20, 21] (therefore the errors that have been committed in the literature, even after going through the whole process of regularization/renormalization in a seemingly clean way). Fourth, we know of no mathematically sound prescription in order to choose the good regularized answer for the determinant, in general. Maybe a better answer to this issue may be given, but it will require further investigation.

3.7. On Determinants

Many fundamental calculations of QFT reduce, in essence, to the computation of the determinant of some suitable operator: at one-loop order, any such theory reduces in fact to a theory of determinants. The operators involved are pseudodifferential (Ψ DO), in loose terms ‘some analytic functions of differential operators’ (such as $\sqrt{1+D}$ or $\log(1+D)$, but *not* $\log D$). This is explained in detail in [22]. It is surprising that this seems not to be a main subject of study among mathematicians, in particular the determinants that involve in its definition some kind of regularization (related to operators that are not trace-class). This piece of calculus falls outside the scope of the standard disciplines and even many physically oriented mathematicians know little about this. The subject has many things in common with divergent series but lacks any reference comparable to the book of Hardy [23]. Actually, this question was already addressed by Weierstrass in a way not without problems, since it leads to non-local contributions that cannot be given a physical meaning in QFT. For completion, let us mention the well established theories of determinants for degenerate operators, for trace-class operators in the Hilbert space, Fredholm operators, etc. [24]

3.8. The Chowla-Selberg Expansion Formula: Basic Aspects

From Jacobi's identity for the θ -function

$$\theta_3(z, \tau) := 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos(2nz), \quad q := e^{i\pi\tau}, \quad \tau \in \mathbb{C} \quad (22)$$

with

$$\theta_3(z, \tau) = \frac{1}{\sqrt{-i\tau}} e^{z^2/i\pi\tau} \theta_3\left(\frac{z}{\tau} \middle| \frac{-1}{\tau}\right), \quad (23)$$

or equivalently

$$\sum_{n=-\infty}^{\infty} e^{-(n+z)^2 t} = \sqrt{\frac{\pi}{t}} \sum_{n=0}^{\infty} e^{-\frac{\pi^2 n^2}{t}} \cos(2\pi n z), \quad z, t \in \mathbb{C}, \quad \operatorname{Re} t > 0. \quad (24)$$

In higher dimensions the relevant expression is Poisson's summation formula, profusely used by Riemann in his original papers (for recent references see [25], namely

$$\sum_{\vec{n} \in \mathbb{Z}^p} f(\vec{n}) = \sum_{\vec{m} \in \mathbb{Z}^p} \tilde{f}(\vec{m}), \quad (25)$$

being \tilde{f} the Fourier transform of f . An important extension of this theory has consisted in the introduction of *truncated sums* since then neither of these fundamental identities is directly applicable [26]. Useful results have been obtained also in these cases, which are very important in physical applications, in terms of *asymptotic series*.

3.8.1. Extended CS formulas (ECS). Consider the zeta function (with $\operatorname{Re} s > p/2$, $A > 0$, $\operatorname{Re} q > 0$)

$$\zeta_{A, \vec{c}, q}(s) = \sum'_{\vec{n} \in \mathbb{Z}^p} \left[\frac{1}{2} (\vec{n} + \vec{c})^T A (\vec{n} + \vec{c}) + q \right]^{-s} = \sum'_{\vec{n} \in \mathbb{Z}^p} [Q(\vec{n} + \vec{c}) + q]^{-s} \quad (26)$$

where the prime indicates that the point $\vec{n} = \vec{0}$ is to be excluded from the sum (an inescapable condition when $c_1 = \dots = c_p = q = 0$). We can write

$$Q(\vec{n} + \vec{c}) + q = Q(\vec{n}) + L(\vec{n}) + \bar{q}. \quad (27)$$

3.8.2. Case $q \neq 0$ ($\operatorname{Re} q > 0$). Then

$$\begin{aligned} \zeta_{A, \vec{c}, q}(s) &= \frac{(2\pi)^{p/2} q^{p/2-s}}{\sqrt{\det A}} \frac{\Gamma(s - p/2)}{\Gamma(s)} + \frac{2^{s/2+p/4+2} \pi^s q^{-s/2+p/4}}{\sqrt{\det A} \Gamma(s)} \\ &\times \sum'_{\vec{m} \in \mathbb{Z}_{1/2}^p} \cos(2\pi \vec{m} \cdot \vec{c}) (\vec{m}^T A^{-1} \vec{m})^{s/2-p/4} K_{p/2-s} \left(2\pi \sqrt{2q \vec{m}^T A^{-1} \vec{m}} \right), \end{aligned} \quad (28)$$

an original expression that we have labeled as [ECS1]. After detailed inspection, it is easy to see here that the pole at $s = p/2$, and its corresponding residue

$$\operatorname{Res}_{s=p/2} \zeta_{A, \vec{c}, q}(s) = \frac{(2\pi)^{p/2}}{\Gamma(p/2)} (\det A)^{-1/2}, \quad (29)$$

are explicitly given in the formula, which has in all the following properties.

- (i) It yields the (analytical continuation of) the multidimensional zeta function in terms of an *exponentially convergent* multiseriess, valid in the *whole* complex plane
- (ii) It exhibits singularities (*simple poles*) of the meromorphic continuation—with the corresponding *residua*—*explicitly*.
- (iii) The only condition on the matrix, A , is that it must correspond to a (*non negative*) *quadratic form*, Q . The vector \vec{c} is *arbitrary*, while q is (to start) any non-negative constant.
- (iv) K_ν is the modified Bessel function of the second kind and the subindex in $\mathbb{Z}_{1/2}^p$ means that only *half* of the vectors $\vec{m} \in \mathbb{Z}^p$ participate in the sum. E.g., if we take an index $\vec{m} \in \mathbb{Z}^p$ we must then exclude $-\vec{m}$, a simple criterion being: one may select those vectors in $\mathbb{Z}^p \setminus \{\vec{0}\}$ whose *first non-zero component is positive*.

3.8.3. *Case* $c_1 = \dots = c_p = q = 0$. This case is a true extension of CS; we will here consider the diagonal subcase only [27]

$$\zeta_{A_p}(s) = \frac{2^{1+s}}{\Gamma(s)} \sum_{j=0}^{p-1} (\det A_j)^{-1/2} \left[\pi^{j/2} a_{p-j}^{j/2-s} \Gamma\left(s - \frac{j}{2}\right) \zeta_R(2s - j) \right. \\ \left. + 4\pi^s a_{p-j}^{\frac{j}{4}-\frac{s}{2}} \sum_{n=1}^{\infty} \sum'_{\vec{m}_j \in \mathbb{Z}^j} n^{j/2-s} (\vec{m}_j^t A_j^{-1} \vec{m}_j)^{s/2-j/4} K_{j/2-s} \left(2\pi n \sqrt{a_{p-j} \vec{m}_j^t A_j^{-1} \vec{m}_j} \right) \right], \quad (30)$$

an expression that truly extends the CS formula and we have labeled as [ECS3d] [27].

4. On Zeta Function Regularization

4.1. Some Considerations on Zeta Regularization

Regularization and renormalization procedures are essential issues in contemporary physics [13]. Among the different methods, zeta function regularization—obtained by analytic continuation in the complex plane of the zeta function of the relevant physical operator in each case—is one of the most beautiful of all. Use of this procedure yields the vacuum energy corresponding to a quantum physical system, with constraints of very different nature. The case of moving boundaries seems to present quite severe difficulties, though some promising approach to deal with them has appeared [28]. Let the Hamiltonian operator, H , of our quantum system to have a spectral decomposition of the form (think as simplest case in a quantum harmonic oscillator): $\{\lambda_i, \varphi_i\}_{i \in I}$, with I some set of indices (it can be discrete, continuous, mixed, or multiple). The quantum vacuum energy is obtained as follows [29]

$$E/\mu = \sum_{i \in I} \langle \varphi_i, (H/\mu) \varphi_i \rangle = \text{Tr}_\zeta H/\mu = \sum_{i \in I} (\lambda_i/\mu)^{-s} \Big|_{s=-1} = \zeta_{H/\mu}(-1), \quad (31)$$

where ζ_A is the zeta function corresponding to the operator A , and the equalities are in the sense of analytic continuation (since, generically, the Hamiltonian operator will not be of the trace class). Actually, this ζ -trace is *no trace in the usual sense*. It is highly

non-linear, as often explained by the author [30]. Some colleagues are however unaware of this fact, which has lead to very serious mistakes and erroneous conclusions in the literature.

The formal sum over the eigenvalues is usually ill defined and the last step involves analytic continuation, inherent with the definition of the zeta function itself. Also, an unavoidable renormalization parameter, μ , with the dimensions of mass, appears in the process, in order to render the eigenvalues of the resulting operator dimensionless, so that the corresponding zeta function can actually be defined. For lack of space, we shall not discuss those basic details here, which are at the starting point of the whole renormalization procedure. The mathematically simple-looking relations above involve deep physical concepts, no wonder that understanding them has taken several decades in the recent history of quantum field theory.

4.2. On the Zero Point Energy and the Casimir Force

In an ordinary QFT, one cannot give a meaning to the *absolute* value of the zero-point energy, and any physically measurable effect comes as an energy *difference* between two situations, such as a quantum field satisfying BCs on some surface as compared with the same in its absence, or one in curved space as compared with the same field in flat space, etc. This difference is the Casimir energy: $E_C = E_0^{BC} - E_0 = \frac{1}{2} (\text{tr } H^{BC} - \text{tr } H)$. But here a problem appears. Imposing mathematical boundary conditions (BCs) on physical quantum fields turns out to be a highly non-trivial issue. This was discussed in detail in a paper by Deutsch and Candelas [31]. These authors quantized em and scalar fields in the region near an arbitrary smooth boundary, and calculated the renormalized vacuum expectation value of the stress-energy tensor, to find out that the energy density diverges as the boundary is approached. Therefore, regularization and renormalization did not seem to cure the problem with infinities in this case and an infinite *physical* energy was obtained if the mathematical BCs were to be fulfilled. However, the authors argued that surfaces have non-zero depth, and its value could be taken as a handy dimensional cutoff in order to regularize the infinities. Just two years after Deutsch and Candelas' work, Kurt Symanzik carried out a rigorous analysis of QFT in the presence of boundaries [32]. Prescribing the value of the quantum field on a boundary means using the Schrödinger representation, and Symanzik was able to show rigorously that such representation exists to all orders in the perturbative expansion. He showed also that the field operator being diagonalized in a smooth hypersurface differs from the usual renormalized one by a factor that diverges logarithmically when the distance to the hypersurface goes to zero. This requires a precise limiting procedure and point splitting to be applied. In any case, the issue was proven by him to be perfectly meaningful within the domains of renormalized QFT. In this case the BCs and the hypersurfaces themselves were treated at a pure mathematical level (zero depth) by using Dirac delta functions.

Not long ago, a new approach to the problem has been postulated [33]. BCs on a

field, ϕ , are enforced on a surface, S , by introducing a scalar potential, σ , of Gaussian shape living on and near the surface. When the Gaussian becomes a delta function, the BCs (Dirichlet here) are enforced: the delta-shaped potential kills *all* the modes of ϕ at the surface. For the rest, the quantum system undergoes a full-fledged QFT renormalization, as in the case of Symanzik's approach. The results obtained confirm those of [31] in the several models studied albeit they do not seem to agree with those of [32]. They seem to be also in contradiction with the ones quoted in the usual textbooks and review articles dealing with the Casimir effect [34], where no infinite energy density when approaching the Casimir plates has been reported. This has been extended by the author using methods of Hadamard regularization, what seems to be a new important development in this direction [35].

5. Quantum Vacuum Fluctuations, Zeta Regularization, and the Cosmological Constant

5.1. Vacuum Energy Fluctuations and the Cosmological Constant

The issue of the cc has got renewed thrust from the observational evidence of an acceleration in the expansion of our Universe, initially reported by two different groups [36]. There was some controversy on the reliability of the results obtained from those observations and on its precise interpretation, but after new data was gathered, there is now consensus among the community of cosmologists that, in fact, an acceleration is there, and that it has the order of magnitude obtained in the above mentioned observations [37, 38, 39]. As a consequence, many theoreticians have urged to try to explain this fact, and also to try to reproduce the precise value of the cc coming from these observations [40, 41, 42].

As crudely stated by Weinberg [43], it is more difficult to explain why the cc is so small but non-zero, than to build theoretical models where it exactly vanishes [44]. Rigorous calculations performed in quantum field theory on the vacuum energy density, ρ_V , corresponding to quantum fluctuations of the fields we observe in nature, lead to values that are many orders of magnitude in excess of those allowed by observations of the space-time around us. Energy always gravitates [45], therefore the energy density of the vacuum, more precisely, the vacuum expectation value of the stress-energy tensor $\langle T_{\mu\nu} \rangle \equiv -\mathcal{E}g_{\mu\nu}$ appears on the rhs of Einstein's equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -8\pi G(\tilde{T}_{\mu\nu} - \mathcal{E}g_{\mu\nu}). \quad (32)$$

It affects cosmology: $\tilde{T}_{\mu\nu}$ contains excitations above the vacuum, and is equivalent to a cc $\Lambda = 8\pi G\mathcal{E}$. Recent observations yield [46]

$$\Lambda_{\text{obs}} = (2.14 \pm 0.13 \times 10^{-3} \text{ eV})^4 \sim 4.32 \times 10^{-9} \text{ erg/cm}^3$$

It is an old idea that the cc gets contributions from zero point fluctuations [47]

$$E_0 = \frac{\hbar c}{2} \sum_n \omega_n, \quad \omega = k^2 + m^2/\hbar^2, \quad k = 2\pi/\Lambda. \quad (33)$$

Evaluating in a box and putting a cut-off at maximum k_{max} corresponding to reliable QFT physics (e.g., the Planck energy)

$$\rho \sim \frac{\hbar k_{\text{Planck}}^4}{16\pi^2} \sim 10^{123} \rho_{\text{obs}}. \quad (34)$$

Assuming one will be able to prove (in the future) that the ground value of the cc is *zero* (as many suspected until recently), we will be left with this *incremental value* coming from the topology or BCs. This sort of two-step approach to the cc is becoming more and more popular recently as a way to try to solve this very difficult issue [48]. We have then to see, using different examples, if this value acquires the correct order of magnitude—corresponding to the one coming from the observed acceleration in the expansion of our universe—under some reasonable conditions. We pursue a quite simple and primitive idea, related with the *global* topology of the universe [49] and in connection with the possibility that a faint scalar field pervading the universe could exist. Fields of this kind are ubiquitous in inflationary models, quintessence theories, and the like. In other words, we do not pretend to solve the old problem of the cc, not even to contribute significantly to its understanding, but just to present simple and usual models which show that the right order of magnitude of (some contributions to) ρ_V which lie in the precise range deduced from the astrophysical observations are not difficult to get. In different words, we only address here the ‘second stage’ of what has been termed by Weinberg [43] the *new* cc problem.

5.2. Vacuum Energy Contribution in Different Models

5.2.1. Simple model with large and small compactified dimensions. We assume the existence of a scalar field extending through the universe and calculate the contribution to the cc from the Casimir energy density of this field, for some typical boundary conditions. Ultraviolet contributions will be set to zero by some mechanism of a fundamental theory. We assume the existence of both large and small dimensions (the total number of large spatial coordinates being always three), some of which may be compactified, so that the global topology of the universe may play an important role [49, 50, 51, 52, 53]. We know [29] that the range of orders of magnitude of the vacuum energy density for common possibilities is not widespread (may only differ by a couple of digits) and one can deal with two simple situations: a scalar field with periodic BCs or spherically compactified [54, 55]). The contribution of the vacuum energy of a small-mass scalar field, conformally coupled to gravity, and coming from the compactification of some small (2 or 3) and some large (1 or 2) dimensions—with compactification radii of the order of 10 to 1000 the Planck length in the first case and of the order of the present radius of the universe, in the second—lead to values that compare well with observational data, in order of magnitude, but with the wrong *sign*.

5.2.2. Braneworld models. An important issue in all the previous analysis is the specific *sign* of the resulting force. For scalar fields and the usual compactifications or BCs

it is impossible to get the right sign corresponding to the accelerated expansion of the universe. However, in braneworld models and others involving supergravitons and fermion fields we have been able to prove that the appropriate sign can be obtained under quite natural conditions.

Braneworld theories may hopefully solve both the hierarchy problem and the cc problem. The bulk Casimir effect can play an important role in the construction (radion stabilization) of braneworlds. We have calculated the bulk Casimir effect (effective potential) for conformal and for massive scalar fields [56]. The bulk is a 5-dim AdS or dS space, with 2 (or 1) 4-dim dS branes (our universe). The results obtained are quite consistent with observational data. A difficulty in this case, however, is the comparison of the vacuum energy density obtained in five dimension with the one corresponding to four dimensions. Even more, six dimensional models are very on fashion now and problems of this kind pop up there too [57].

5.2.3. Supergraviton theories We have also computed the effective potential for some multi-graviton models with supersymmetry [58]. In one case, the bulk is a flat manifold with the torus topology $\mathbb{R} \times \mathbb{T}^3$, and it can be shown that the induced cc can be rendered *positive* due to topological contributions [59]. Previously, the case of \mathbb{R}^4 had been considered. In the multi-graviton model the induced cc can indeed be positive, but only if the number of massive gravitons is sufficiently large, what is not easy to fit in a natural way. In the supersymmetric case, however, the cc turns out to be positive just by imposing anti-periodic BC in the fermionic sector. An essential issue in our model is to allow for non-nearest-neighbor couplings.

For the torus topology we have got the topological contributions to the effective potential to have always a fixed sign, which depends on the BC one imposes. They are negative for periodic fields, and positive for anti-periodic ones. But topology provides then a mechanism which, in a natural way, permits to have a positive cc in the multi-supergravity model with anti-periodic fermions. The value of the cc is regulated by the corresponding size of the torus. We can most naturally use the minimum number, $N = 3$, of copies of bosons and fermions, and show that —as in the first, much more simple example, but now with the right sign!— within our model the observational values for the cc can be approximately matched, by making quite reasonable adjustments of the parameters involved. As a byproduct, the results that we have obtained [59] might also be relevant in the study of electroweak symmetry breaking in models with similar type of couplings, for the deconstruction issue.

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